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Toroid moments in the momentum and angular momentum loss by a radiating arbitrary source

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In the context of experimental evidence concerning a nuclear toroid dipole, we briefly present here an exact but tedious calculation of the angular momentum loss, recoil force, and radiation intensity for an arbitrary source in terms of electric, magnetic, and toroid multipoles, thus emphasizing the importance of the latter in getting the results in closed forms, unbiased by approximations. Corrections to some familiar formulas from books, mostly on account of time varying toroid moments, are found and discussed.

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In a previous paper [1] the angular momentum loss through radiation of electromagnetic waves by a pointlike toroidal dipole has been calculated and discussed. The toroid dipole, introduced by Zeldovich [2] led to the theoretical construction [3] of a whole class of toroid multipolar moments and distributions shown to be indispensable for a correct and complete multipole characterization of an arbitrary source [4]. Over the years the activity in the domain of toroid moments increased [3,4,5] and we note now experimental evidence of a nuclear spin-dependent contribution to atomic parity nonconservation claimed to agree with that predicted to arise from a nuclear toroid dipole [6].

In this context, here we present a short account on a thorough investigation aimed to find an exact expression for the angular momentum loss, recoil force, and radiation intensity for the most general type of source (which includes toroidal structures, previously either not considered or unproperly treated) described by the charge density $\rho(\vec{r},t)$ and the current density $\vec{j}(\vec{r},t)$ satisfying only the continuity relation $\partial\rho/\partial t+\nabla\cdot\vec{j}(\vec{r},t)=0$, with the time dependence left arbitrary. As it is well known (see, e.g., Ref. [7]) the rate of angular momentum loss through radiation is given by

$$\frac{d\vec{M}}{dt}=\lim_{R_0\to\infty}\frac{R_0^3}{4\pi}\int d\Omega[(\vec{n}\cdot\vec{E})(\vec{n}\times\vec{E})+(\vec{n}\cdot\vec{B})(\vec{n}\times\vec{B})], \tag{1}$$

the recoil force (i.e., the rate of momentum loss) by

$$\vec{F}=\lim_{R_0\to\infty}\left[-\frac{R_0^2}{4\pi}\int d\Omega\left(\vec{E}(\vec{E}\cdot\vec{n})+\vec{B}(\vec{B}\cdot\vec{n})-\vec{n}\frac{(\vec{E}^2+\vec{B}^2)}{2}\right)\right], \tag{2}$$

and the radiation intensity by

$$I=\lim_{R_0\to\infty}\frac{cR_0^2}{4\pi}\int d\Omega(\vec{E}\times\vec{H})\cdot\vec{n}, \tag{3}$$

where $\vec{n}$ is the unit vector pointing in the direction of $\vec{R}_0$, the integration is over a spherical surface of large radius R_0 while the fields $\vec{E}$, $\vec{B}$ are calculated in terms of ρ, $\vec{j}$ as usually through the retarded scalar and vector potentials:

$$\vec{E}(\vec{r},t)=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi(\vec{r},t),\quad \vec{H}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t),$$

$$\varphi(\vec{r},t)=\int d^3\vec{r}\,'\frac{\rho\left(\vec{r}\,',t-\left|\frac{\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|},$$

$$\vec{A}(\vec{r},t)=\frac{1}{c}\int d^3\vec{r}\,'\frac{\vec{j}\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}. \tag{4}$$

Our task is to express without any approximations $d\vec{M}/dt$, $\vec{F}$, and I in terms of the electric $\overline{r_{lm}^{2n}}(t)$, magnetic $\overline{\rho_{lm}^{2n}}(t)$, and toroid $\overline{R_{lm}^{2n}}(t)$ mean square radii of order n and multipolarity l which, as shown in Ref. [3], realize a correct multipolar decomposition of the source and describe it just to the same extent as do $\rho(\vec{r},t)$, $\vec{j}(\vec{r},t)$:

$$\overline{r_{lm}^{2n}}(t)=\frac{\sqrt{4\pi}}{\sqrt{2l+1}}\int r^{l+2n}Y_{lm}^*(\vec{n})\rho(\vec{r},t)d^3\vec{r},\vec{n}=\frac{\vec{r}}{r}, \tag{5}$$

$$\overline{\rho_{lm}^{2n}}(t)=-\frac{i}{c}\frac{\sqrt{4\pi l}}{\sqrt{(l+1)(2l+1)}}\int r^{2n+l}\vec{Y}_{llm}^*(\vec{n})\cdot\vec{j}(\vec{r},t)d^3r, \tag{6}$$

$$\overline{R_{lm}^{2n}}(t)=\frac{-1}{c(2l+1)}\sqrt{\frac{4\pi l}{l+1}}\int d^3r r^{l+2n+1}\times\left(\frac{\sqrt{l}}{(2l+2n+3)}\vec{Y}_{ll+1m}^*(\vec{n})+\frac{\sqrt{l+1}}{2(n+1)}\vec{Y}_{ll-1m}^*(\vec{n})\right)\cdot\vec{j}(\vec{r},t). \tag{7}$$

($\vec{Y}_{ll'm}$ are the usual vector spherical harmonics, $l=1,2,...$; $m=-l,\ldots,+l$ and $n=0,1,2,\ldots$.) For zero order ($n=0$), Eqs. (5), (6), and (7) furnish the system's electric, magnetic, and toroid multipole moments themselves. To get exact expressions for $\vec{F}$ and I, one needs the correct large distances (r)

behavior of the fields $\vec{E}$, $\vec{B}$ to order $O(1/r)$, while for $d\vec{M}/dt$ one needs the leading $O(1/r^2)$ behavior of $(\vec{n}\cdot\vec{E})$, $(\vec{n}\cdot\vec{H})$, since to order $O(1/r)$ these quantities are obviously zero due to the transversality of the fields in the wave zone. The resulting formulas are quite long, but exact. They will be presented in detail elsewhere [8]. Nevertheless we list here the closed form expression for $d\vec{M}/dt$ which amounts to an exhaustive generalization of the result given in Ref. [1] in the toroid dipole case:

$$\begin{aligned}\frac{dM_\mu}{dt}=&\sum_{l,m}\frac{(-1)^{m+1}}{(2l-1)!!(2l+1)!!}\frac{1}{c^{2l}}\\&\times\left(\frac{l+2}{2l+1}\frac{\sqrt{l+1}}{\sqrt{2l+3}}C\begin{matrix}l+1 & 1 & l\\ m+\mu & -\mu & m\end{matrix}A\right.\\&+(l+1)\sqrt{l(2l-1)}C\begin{matrix}l-1 & 1 & l\\ m+\mu & -\mu & m\end{matrix}B\\&\left.+\frac{i}{c}(l+1)\frac{\sqrt{l+1}}{\sqrt{l}}C\begin{matrix}l & 1 & l\\ m+\mu & -\mu & m\end{matrix}C\right),\end{aligned}\tag{8}$$

where A, B, C multiplying the Clebsch-Gordan coefficients stand for

$$\begin{aligned}A=&\sum_{n'}\frac{1}{c^{2n'+2}n'!}(Q_{lm}^{(0)(l)}M_{l+1,-\mu-m}^{(n')(l+2n'+2)}\\&-M_{lm}^{(n')(l+2n')}Q_{l+1,-m-\mu}^{(0)(l+2)})\\&+\sum_{n,n'}\frac{1}{c^{2n+2n'+3}n!n'!}(M_{lm}^{(n')(l+2n')}T_{l+1,-m-\mu}^{(n)(l+2n+3)}\\&-T_{lm}^{(n)(l+2n+1)}M_{l+1,-\mu-m}^{(n')(l+2n'+2)}),\end{aligned}\tag{9}$$

$$\begin{aligned}B=&\sum_{n'}\frac{1}{c^{2n'}n'!}(Q_{lm}^{(0)(l)}M_{l-1,-\mu-m}^{(n')(l+2n')}\\&-M_{lm}^{(n')(l+2n')}Q_{l-1,-m-\mu}^{(0)(l)})\\&+\sum_{n,n'}\frac{1}{c^{2n+2n'+1}n!n'!}(M_{lm}^{(n')(l+2n')}T_{l-1,-m-\mu}^{(n)(l+2n+1)}\\&-T_{lm}^{(n)(l+2n+1)}M_{l-1,-\mu-m}^{(n')(l+2n')}),\end{aligned}\tag{10}$$

$$\begin{aligned}C=&-Q_{lm}^{(0)(l)}Q_{l,-\mu-m}^{(0)(l+1)}+\sum_{n}\frac{1}{c^{2n+1}n!}(Q_{lm}^{(0)(l)}T_{l,-\mu-m}^{(n)(l+2n+2)}\\&+T_{lm}^{(n)(l+2n+1)}Q_{l,-\mu-m}^{(0)(l+1)})\\&-\sum_{n,n'}\frac{1}{c^{2n+2n'}n!n'!}(M_{lm}^{(n)(l+2n)}M_{l,-\mu-m}^{(n')(l+2n'+1)}\\&+\frac{1}{c^2}T_{lm}^{(n)(l+2n+1)}T_{l,-\mu-m}^{(n')(l+2n'+2)}).\end{aligned}\tag{11}$$

All $Q_{lm}^{(n)(\nu)}$, $M_{lm}^{(n)(\nu)}$, $T_{lm}^{(n)(\nu)}$ in Eqs. (9)–(11) are, as usual, functions of the retarded time $t-(r/c)$ and represent up to a known numerical factor time derivatives of the mean square radii of various orders:

$$\begin{pmatrix}Q_{lm}^{(n)(\nu)}\left(t-\frac{r}{c}\right)\\ M_{lm}^{(n)(\nu)}\left(t-\frac{r}{c}\right)\\ T_{lm}^{(n)(\nu)}\left(t-\frac{r}{c}\right)\end{pmatrix}=\frac{(2l+1)!!}{2^n(2l+2n+1)!!}\frac{d^\nu}{dt^\nu}\begin{pmatrix}\overline{r_{lm}^{2n}}\left(t-\frac{r}{c}\right)\\ \overline{\rho_{lm}^{2n}}\left(t-\frac{r}{c}\right)\\ \overline{R_{lm}^{2n}}\left(t-\frac{r}{c}\right)\end{pmatrix}.\tag{12}$$

The result for dM_μ/dt is given in spherical basis $[\mu=-,0,+;M_{(+)}=-(1/\sqrt{2})(M_x+iM_y)$, $M_{(-)}=(1/\sqrt{2})\times(M_x-iM_y)$, $M_{(0)}=M_z]$. Retaining from the closed formula above the contributions of the first multipoles, we get for $d\vec{M}/dt$ to the order $1/c^5$ inclusively:

$$\begin{aligned}\frac{dM_\alpha}{dt}=&-\frac{2}{3c^3}\varepsilon_{\alpha\beta\gamma}(\dot{d}_\beta\ddot{d}_\gamma+\dot{m}_\beta\ddot{m}_\gamma)\\&+\frac{1}{c^4}\left(\frac{1}{5}(-\dddot{m}_{\alpha\beta}\dot{d}_\beta-\ddot{m}_{\alpha\beta}\ddot{d}_\beta+2\dddot{Q}_{\alpha\beta}\dot{m}_\beta+2\ddot{Q}_{\alpha\beta}\ddot{m}_\beta)\right.\\&\left.+\frac{2}{3}\varepsilon_{\alpha\beta\gamma}(\dot{d}_\beta\dddot{t}_\gamma+\ddot{t}_\beta\ddot{d}_\gamma)\right)-\frac{1}{5c^5}\left[\frac{1}{3}(2\ddddot{t}_{\alpha\beta}\dot{m}_\beta\right.\\&+2\dddot{t}_{\alpha\beta}\ddot{m}_\beta-3\dddot{m}_{\alpha\beta}\ddot{t}_\beta-3\ddot{m}_{\alpha\beta}\dddot{t}_\beta)+\varepsilon_{\alpha\beta\gamma}\left(2\ddot{Q}_{\delta\beta}\dddot{Q}_{\delta\gamma}\right.\\&\left.\left.+\frac{10}{3}\ddot{t}_\beta\dddot{t}_\gamma+\frac{1}{3}\dot{m}_\beta\ddddot{\overline{\rho_\gamma^2}}-\frac{1}{3}\ddot{m}_\beta\dddot{\overline{\rho_\gamma^2}}+\frac{1}{2}\ddot{m}_{\delta\beta}\dddot{m}_{\delta\gamma}\right)\right]\\&+\left(\text{higher than }\frac{1}{c^5}\text{ terms}\right).\end{aligned}\tag{13}$$

Summation over repeated Cartesian indices is understood; dots mean time derivatives and the argument is of course $t-(r/c)$. To the same $1/c^5$ order, using our exact results for the recoil force and radiation intensity, we have found

$$F_\alpha = -\frac{2}{3c^4}\varepsilon_{\alpha\beta\gamma}\ddot{d}_\beta \ddot{m}_\gamma - \frac{1}{5c^5}\ddot{m}_{\alpha\beta}\ddot{m}_\beta - \frac{2}{5c^5}\dddot{Q}_{\alpha\beta}\ddot{d}_b - \frac{2}{3c^5}\varepsilon_{\alpha\beta\gamma}\ddot{m}_\beta \dddot{t}_\gamma + \cdots, \tag{14}$$

$$I = \frac{2}{3c^3}\ddot{\vec{d}}^2 + \frac{2}{3c^3}\ddot{\vec{m}}^2 - \frac{4}{3c^4}\ddot{\vec{d}}\dddot{\vec{t}} + \frac{2}{3c^5}\dddot{\vec{t}}^2 + \frac{2}{15c^5}\ddot{\vec{m}}\ddddot{\vec{\rho^2}} + \frac{1}{5c^5}\dddot{Q}_{\alpha\beta}\dddot{Q}_{\alpha\beta} + \frac{1}{20c^5}\dddot{m}_{\alpha\beta}\dddot{m}_{\alpha\beta} + \cdots. \tag{15}$$

In Eqs. (13), (14), and (15), $\vec{d}$ and $\vec{m}$ are the usual electric and magnetic dipole moments, $\vec{t}$ is the toroid dipole $\vec{t} = (1/10c)\int[\vec{r}(\vec{r}\cdot\vec{j}) - 2\vec{r}^2\vec{j}]d^3r$, $\overline{\rho_\gamma^2}$ is the first mean square radius of the magnetic dipole distribution $\overline{\rho_\gamma^2} = (1/2c)\varepsilon_{\gamma\alpha\beta}\int r_\alpha r^2 j_\beta d^3r$, $Q_{\alpha\beta}$, $m_{\alpha\beta}$, and $t_{\alpha\beta}$, are the electric, magnetic, and toroid quadrupoles: $Q_{\alpha\beta} = (1/2)\int\rho(\vec{r},t)(x_\alpha x_\beta - 1/3\delta_{\alpha\beta}r^2)d^3r$, $M_{\alpha\beta} = (1/3c)\int[(\vec{r}\times\vec{j})_\alpha x_\beta + (\vec{r}\times\vec{j})_\beta x_\alpha]d^3r$, $t_{\alpha\beta} = (1/28c)\int[4x_\alpha x_\beta(\vec{r}\cdot\vec{j}) - 5r^2(x_\alpha j_\beta + x_\beta j_\alpha) + 2r^2(\vec{r}\cdot\vec{j})\delta_{\alpha\beta}]d^3r$.

At this point, we note that our expression Eq. (14) for the recoil force includes all terms to the order $1/c^5$ inclusively, while the familiar one calculated in Ref. [7] (in problem 2 at the end of paragraph 71, the Russian 1988 edition), working with the fields correct only to the $1/c^3$ order is

$$F_\alpha^{(\text{ref. [7]})} = -\frac{1}{c^4}\left(\frac{1}{15c}\dddot{D}_{\alpha\beta}\ddot{d}_\beta + \frac{2}{3}(\ddot{\vec{d}}\times\ddot{\vec{m}})_\alpha\right) \tag{16}$$

(our $Q_{\alpha\beta}$ is $1/6D_{\alpha\beta}$ of Ref. [7]) and contains only the first and the third terms of our Eq. (14), while the second and the fourth terms (both of order $1/c^5$, the fourth one being a toroid contribution) are missing. The last (fourth) term in our Eq. (14) amounts to a contribution to the recoil force coming from the less usual toroid dipole moment:

$$\vec{F}^{\text{magnetic-toroid}} = -\frac{2}{3c^5}\ddot{\vec{m}}\times\dddot{\vec{t}}, \tag{17}$$

which, being of order $(1/c^5)$, is to be considered on the same footing with the third $(1/c^5)$ term of Eq. (14) (computed in Ref. [7]) and the second $(1/c^5)$ term in Eq. (14) (missed in Ref. [7]). We also note that in the expression of the angular momentum loss $d\vec{M}/dt$ calculated in Ref. [7] (problem 2 at the end of paragraph 72, the Russian 1988 edition) only the first term of our Eq. (13) is given, while Eq. (13) completes the result given in Ref. [7] with the remaining contributions of order $1/c^3$ (of magnetic type, probably already known) of order $1/c^4$ (which include a toroid dipole piece) and of order $1/c^5$ (which also include toroid pieces, but besides the toroid dipole moment, this time the toroid quadrupole moment also contributes).

In conclusion, we have shown that for the most general configuration of charges and currents (including toroidal current structures, usually partly lost through approximations) the radiation intensity as well as the rates of momentum and angular momentum loss through radiation of electromagnetic waves, within classical electrodynamics, can be calculated exactly for any time dependence of the sources. We did that calculation with the aid of the multipole decomposition of Ref. [3] in which a third class of multipoles, the toroid ones, was introduced and shown to be indispensable in giving a full description of the source. The results are expressed in terms of time derivatives of the system's electric multipole moments and magnetic and toroid mean square radii (of various orders) whose time dependence is left arbitrary. Inclusion of the toroid multipoles turns out to be compulsory if one wants to get results unbiased by approximations. By retaining only the first multipoles in the exact results obtained, corrections to some familiar formulas were found on account of e.g., time-varying toroid moments.

Note that our analysis goes far beyound those from textbooks since our fields are exact everywhere outside the sources and correctly parametrized in terms of the latter. At the end of Chap. 7 of Ref. [8] we show explicitly from where the discrepancies between our formulas and the corresponding ones from Ref. [7] come.

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